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RADIATION HEAT TRANSFER THROUGH SCATTERING AND ABSORBING NONISOTHERMAL LAYERS

(NASA TM X-54534)

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SUMMARY

Steady-state radiation heat transfer through layers where both scattering and absorption occur within the layers is treated analytically by means of single dimensional fluxes. The set of simultaneous equations for the flux in the direction of heat flow, the flux in the opposite direction, and a heat balance equation has a general solution to which boundary conditions are applied to derive expressions for desired quantities for an arbitrary layer. In this way the transfer through a layer and the emission from it, as well as its temperature distribution, are derived in terms of the absorption and scattering coefficients of the layer, the index of refraction, the lattice conductivity and the heat applied to it. The treatment includes the effects of surface reflections.

Radiation transfer through nonradiating layers is also treated in order to solve the equations which obtain for the absorption and scattering coefficients so that these can be calculated from optical transmission measurements.

INTRODUCTION

Radiation heat transfer through nonisothermal layers, where both scattering and absorption occur, is a very difficult situation to treat in its full generality. One method is to use electronic data processing machines to arrive at numerical answers for specific situations; however, it is advantageous to obtain analytical expressions since it is usually possible to infer more from such expressions about the mechanisms that occur and the directions to manipulate parameters in order to obtain desired results. It is, however, normally necessary to simplify the situation in order to be able to treat it mathematically. This paper simplifies the actual situation by treating only completely diffuse radiation by a single dimensional heat flux calculation and therefore neglects any three-dimensional effects. Though

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this simplification undoubtedly decreases the accuracy of the results, it allows one to handle rather complicated situations and to obtain useful information about the mechanisms occurring.

SYMBOLS

a absorption coefficient for diffuse radiation

b constant equal to $4\sigma' n^2 T_o^3$

D thickness of layer

E black-body radiant energy flux

I radiant energy flux in the direction of the positive x axis

J radiant energy flux in the direction of the negative x axis

k lattice thermal conductivity

n index of refraction

s scattering coefficient for diffuse radiation

T temperature (absolute)

β optical constant for nonisothermal case equal to $\sigma/(a + 2s)$

β_o optical constant for isothermal case equal to $\sqrt{a/(a + 2s)} = \sigma_o/(a + 2s)$

ϵ emissivity

η flux (energy) gradient at surface $(dE/dx)_{surface}$

κ material constant representing ratio of radiant transfer to lattice transfer in the center of an optically dense layer and equal to $2b/k(a + 2s)$

ρ diffuse reflectance of a layer

ρ_i total diffuse reflectance at an interface where the index of refraction is decreasing

ρ_o total diffuse reflectance at an interface where the index of refraction is increasing

σ optical constant for nonisothermal case similar to an extinction coefficient equal to $\sigma_o \sqrt{(1 + \kappa)}$

σ_o isothermal extinction coefficient equal to $\sqrt{a(a + 2s)}$

σ' Stefan-Boltzmann radiation constant

τ diffuse transmittance of layer

BASIC THEORETICAL ASSUMPTIONS

The theoretical method used in this study is based on a system originally conceived by Schuster (refs. 1 and 2) and added to by Hamaker (ref. 3); the notation used here is essentially that of Hamaker. For the isothermal case the method is equivalent to that developed by Kubelka and Munk and extended by others (refs. 4-8). With suitable changes in notation (see ref. 3) their set of equations can be transformed into the system discussed here and vice versa. The basic method is that of dividing the flux into two parts: one flowing in a positive direction, and the other in a negative direction. A set of simultaneous differential equations is used to describe these fluxes and the other necessary parameters. Since only a forward and a backward flux are considered, this is a one-dimensional calculation and therefore has as a basic assumption that the incident radiation is diffuse (i.e., the intensity is equal for all angles of incidence) and that the radiation scattered sideways is compensated for by an equal contribution from neighboring parts of the layer (i.e., the area investigated is either small in cross section compared with the total illuminated cross section of the sample or is large compared to the thickness of the sample). This condition is not a severe limitation since many practical heat-transfer problems are concerned with diffuse radiation.

The treatment for the situations where temperature gradients are present suffers from the further limitation that only total radiation is considered and therefore the fact that the wave-length distribution of black-body emission changes with temperature is not taken into account. Also, it is assumed that the properties of the material change only gradually. This then implies the assumption that the temperature gradient across the sample which is being measured is small. Practically all the methods of calculation in use today also suffer from this limitation and in practice there are calculation schemes which can alleviate the problem.

ISOTHERMAL LAYERS

General Solutions

The total radiant flux is divided into two parts:

I the flux in the direction of the positive x axis

J the flux in the direction of the negative x axis

An absorption coefficient a is defined by requiring that ($aI\,dx$) be the amount of the radiation absorbed from the flux I on passing through an infinitesimal layer dx; a scattering coefficient s is similarly defined by requiring that the flux scattered backward from I (and therefore added to J) in an infinitesimal layer dx is ($sI\,dx$). On passing through this layer, I will then be diminished by the amount absorbed and the amount scattered, but will be increased by the flux lost by scattering from J or:

$$dI/dx = -(a + s)I + sJ \tag{1}$$

Similarly,

$$dJ/dx = (a + s)J - sI \tag{2}$$

The general solutions of these equations can be found by putting

$$I = C_1e^{\sigma x} + C_2e^{-\sigma x} \tag{3}$$

$$J = C_3e^{\sigma x} + C_4e^{-\sigma x} \tag{4}$$

only two of the four constants $C_1 \ldots C_4$ being arbitrary. The solutions (using the same notation as Hamaker) are then:

$$I = A(1 - \beta_o)e^{\sigma_o x} + B(1 + \beta_o)e^{-\sigma_o x} \tag{5}$$

$$J = A(1 + \beta_o)e^{\sigma_o x} + B(1 - \beta_o)e^{-\sigma_o x} \tag{6}$$

where

$$\sigma_o = \sqrt{a(a + 2s)} \tag{7}$$

$$\beta_o = \sqrt{a/(a + 2s)} = \sigma_o/(a + 2s) \tag{8}$$

both roots being taken with a positive sign. In these equations A and B are constants to be determined by the boundary conditions.

Specific Solutions

One of the cases for which specific solutions are desired is that of a layer placed in a beam of diffuse radiation where there is reflection from both internal and external surfaces.

At an interface where the index of refraction is increasing, let the reflectivity equal ρ_o. At an interface where the index of refraction is decreasing, let the reflectivity equal ρ_i. The former parameter can be calculated from the index of refraction by integrating the fresnel reflection over the solid angle of incidence and dividing by the total radiation. This integration has been carried out by Walsh (ref. 9), and numerical values for the reflectivity as a function of the index of refraction have been calculated and tabulated by Ryde and Cooper (ref.10). At an interface where the index of refraction is decreasing, the reflectivity can be shown to be $[(n^2 - 1)/n] + (\rho_o/n^2)$ where the additional terms are due to the amount of light that is totally reflected. These terms can be an important, though very often neglected, factor in heat-transfer calculations. For instance, for a material with an index of refraction of 1.5, ρ_i would be 0.595; for a material of index of refraction 2, ρ_i would be 0.788, both factors being quite significant.

The following nomenclature will be used (where D is the thickness of a layer):

I_i incident flux at $x = 0$

I_o forward flux immediately inside the interface $x = 0$

J_o backward flux immediately inside the interface $x = 0$

J_D backward flux immediately inside the interface $x = D$

I_D forward flux immediately inside the interface $x = D$

There is assumed to be no incident flux on the back surface $x = D$.

Then the boundary conditions are that at the front surface $x = 0$, part ρ_o of the incident radiation I_i is reflected back, and part $1 - \rho_o$ is transmitted. The flux immediately below this interface I_o is composed of this flux $(1 - \rho_o)I_i$ plus that flux reflected from the inner surface of $x = 0$ or $\rho_i J_o$, or

at $x = 0$:

$$I_o = (1 - \rho_o)I_i + \rho_i J_o \tag{9}$$

At the back surface $x = D$, since there is no incident radiation, the only flux is that part ρ_i reflected from the remaining forward flux I_D or

at $x = D$:

$$J_D = \rho_i I_D \tag{10}$$

substituting in these equations for I_o, J_o, I_D, and J_D from equations (5) and (6) gives:

$$A(1 - \beta_o) + B(1 + \beta_o) = (1 - \rho_o)I_i + \rho_i A(1 + \beta_o) + \rho_i B(1 - \beta_o) \tag{11}$$

and

$$A(1 + \beta_o)e^{\sigma_o D} + B(1 - \beta_o)e^{-\sigma_o D} = \rho_i A(1 - \beta_o)e^{\sigma_o D} + \rho_i B(1 + \beta_o)e^{-\sigma_o D} \tag{12}$$

These are the equations to be solved from the constants A and B for these particular boundary conditions. They are (when the exponentials are substituted for by hyperbolic functions):

$$A = \frac{I_i e^{-\sigma_o D}(1 - \rho_o)[\beta_o(1 + \rho_i) - (1 - \rho_i)]}{2\left\{[\beta_o^2(1 + \rho_i)^2 + (1 - \rho_i)^2]\sinh \sigma_o D + 2\beta_o(1 - \rho_i^2)\cosh \sigma_o D\right\}} \tag{13}$$

and

$$B = \frac{I_i e^{\sigma_o D}(1 - \rho_o)[\beta_o(1 + \rho_i) + (1 - \rho_i)]}{2\left\{[\beta_o^2(1 + \rho_i)^2 + (1 - \rho_i)^2]\sinh \sigma_o D + 2\beta_o(1 - \rho_i^2)\cosh \sigma_o D\right\}} \tag{14}$$

Using these values in equations (5) and (6) gives the following expressions for I_x (the forward flux at x) and J_x (the backward flux at x):

$$I_x = \frac{I_i(1 - \rho_o)\left\{[\beta_o(1 + \rho_i) + (1 - \rho_i)](1 + \beta_o)e^{-\sigma_o x}e^{\sigma_o D} + [\beta_o(1 + \rho_i) - (1 - \rho_i)](1 - \beta_o)e^{\sigma_o x}e^{-\sigma_o D}\right\}}{2\left\{[\beta_o^2(1 + \rho_i)^2 + (1 - \rho_i)^2]\sinh \sigma_o D + 2\beta_o(1 - \rho_i^2)\cosh \sigma_o D\right\}} \tag{15}$$

$$J_x = \frac{\begin{array}{l} I_i(1-\rho_o)\{[\beta_o(1+\rho_i)+(1-\rho_i)](1-\beta_o)e^{-\sigma_o x}e^{\sigma_o D} \\ \quad + [\beta_o(1+\rho_i)-(1-\rho_i)](1+\beta_o)e^{\sigma_o x}e^{-\sigma_o D}\} \end{array}}{2\{[\beta_o^2(1+\rho_i)^2+(1-\rho_i)^2]\sinh \sigma_o D + 2\beta_o(1-\rho_i^2)\cosh \sigma_o D\}} \tag{16}$$

In practice it is impossible to check these quantities experimentally; what can be checked, however, is the transmission and the reflectivity. To arrive at these quantities the forward flux immediately under the back surface I_D is determined by substituting D for x in equation (15). Then

$$I_D = \frac{I_i 2\beta_o(1-\rho_o)}{[\beta_o^2(1+\rho_i)^2+(1-\rho_i)^2]\sinh \sigma_o D + 2\beta_o(1-\rho_i^2)\cosh \sigma_o D} \tag{17}$$

The transmission τ is then the ratio of the amount of radiation of the above that gets through the surface $(1-\rho_i)I_D$ to the incident radiation, or

$$\tau = I_D(1-\rho_i)/I_i \tag{18}$$

giving for the transmission

$$\tau = \frac{2\beta_o(1-\rho_o)(1-\rho_i)}{[\beta_o^2(1+\rho_i)^2+(1-\rho_i)^2]\sinh \sigma_o D + 2\beta_o(1-\rho_i^2)\cosh \sigma_o D} \tag{19}$$

The reflectivity ρ can be found similarly if the fraction of incident radiation reflected from the front surface $\rho_o I_i$ is added to the amount of backward flux that gets through the interface $(1-\rho_i)J_o$. Then

$$\rho = \frac{\begin{array}{l} [(1-\rho_i)^2 - \beta_o^2(1-\rho_i-2\rho_o)(1+\rho_i)]\sinh \sigma_o D \\ \quad + 2\beta_o(\rho_o+\rho_i)(1-\rho_i)\cosh \sigma_o D \end{array}}{[\beta_o^2(1+\rho_i)^2+(1-\rho_i)^2]\sinh \sigma_o D + 2\beta_o(1-\rho_i^2)\cosh \sigma_o D} \tag{20}$$

It is also possible to calculate the absorptivity α of the layer since $\alpha + \rho + \tau = 1$. It is

$$\alpha = \frac{2\beta_o(1-\rho_o)[\beta_o(1+\rho_i)\sinh \sigma_o D + (1-\rho_i)(\cosh \sigma_o D - 1)]}{[\beta_o^2(1+\rho_i)^2+(1-\rho_i)^2]\sinh \sigma_o D + 2\beta_o(1-\rho_i^2)\cosh \sigma_o D} \tag{21}$$

This is also the emission of the layer relative to black-body radiation according to Kirchhoff's law.

Determining Optical Constants From Transmission Measurements

One of the objects of making transmission measurements is to use them to calculate optical constants of the material. In order to do this, equation (19) for the transmission of the material has to be solved for the constants. Cross-multiplying in equation (19) gives:

$$2\beta_o(1 - \rho_o)(1 - \rho_i) = \tau\beta_o{}^2(1 + \rho_i)^2 \sinh \sigma_o D + \tau(1 - \rho_i)^2 \sinh \sigma_o D$$
$$+ 2\beta_o\tau(1 - \rho_i{}^2)\cosh \sigma_o D \tag{22}$$

or, by regrouping the terms,

$$\tau(1 - \rho_i)^2 \sinh \sigma_o D = \beta_o\{2[(1 - \rho_o)(1 - \rho_i) - \tau(1 - \rho_i{}^2)\cosh \sigma_o D]$$
$$- \tau\beta_o(1 + \rho_i)^2 \sinh \sigma_o D\} \tag{23}$$

If two layers of thicknesses D_1 and D_2 and transmissions τ_1 and τ_2, respectively, are considered, then (dividing by equal quantities)

$$\frac{\tau_1(1 - \rho_i)^2 \sinh \sigma_o D_1}{\tau_2(1 - \rho_i)^2 \sinh \sigma_o D_2}$$
$$= \frac{2[(1 - \rho_o)(1 - \rho_i) - \tau_1(1 - \rho_i{}^2)\cosh \sigma_o D_1] - \tau_1\beta_o(1 + \rho_i)^2 \sinh \sigma_o D_1}{2[(1 - \rho_o)(1 - \rho_i) - \tau_2(1 - \rho_i{}^2)\cosh \sigma_o D_2] - \tau_2\beta_o(1 + \rho_i)^2 \sinh \sigma_o D_2}$$

or (again cross-multiplying),

$$2\tau_1(1 - \rho_o)(1 - \rho_i)^3 \sinh \sigma_o D_1$$
$$- 2\tau_1\tau_2(1 + \rho_i)(1 - \rho_i)^3 \sinh \sigma_o D_1 \cosh \sigma_o D_2$$
$$- \beta_o\tau_1\tau_2(1 + \rho_i)^2(1 - \rho_i)^2 \sinh \sigma_o D_1 \sinh \sigma_o D_2$$
$$= 2\tau_2(1 - \rho_o)(1 - \rho_i)^3 \sinh \sigma_o D_2$$
$$- 2\tau_1\tau_2(1 + \rho_i)(1 - \rho_i)^3 \sinh \sigma_o D_2 \cosh \sigma_o D_1$$
$$- \beta_o\tau_1\tau_2(1 + \rho_i)^2(1 - \rho_i)^2 \sinh \sigma_o D_1 \sinh \sigma_o D_2 \tag{25}$$

All the terms involving β_o drop out of the above equation giving (having made use of the identity $\sinh x \cosh y - \cosh x \sinh y \equiv \sinh(x - y)$)

$$\frac{\sinh \sigma_o D_1}{\tau_2} - \frac{\sinh \sigma_o D_2}{\tau_1} = \frac{(1 + \rho_i)}{(1 - \rho_o)} \sinh \sigma_o (D_1 - D_2) \qquad (26)$$

or, if sample thicknesses are chosen such that

$$D_1 = 2D_2 \equiv 2D \qquad (27)$$

$$\frac{\sinh 2\sigma_o D}{\tau_2} - \frac{\sinh \sigma_o D}{\tau_1} = \frac{(1 + \rho_i)}{(1 - \rho_o)} \sinh \sigma_o D \qquad (28)$$

but

$$\sinh 2x \equiv 2 \sinh x \cosh x \qquad (29)$$

and

$$\frac{2 \sinh \sigma_o D \cosh \sigma_o D}{\tau_2} - \frac{\sinh \sigma_o D}{\tau_1} = \frac{(1 + \rho_i)}{(1 - \rho_o)} \sinh \sigma_o D \qquad (30)$$

and

$$\cosh \sigma_o D = \frac{\tau_2[\tau_1(1 + \rho_i) + (1 - \rho_o)]}{2\tau_1(1 - \rho_o)} \qquad (31)$$

allowing one to calculate σ_o from two transmission measurements.

Once σ_o is known, β_o can be found either by solving equation (19) by the quadratic formula, or with an electronic data processing machine.

It should be noted at this point that these equations are only valid for experimental situations where diffuse radiation is incident on a sample and the total hemispherical transmission is measured. The usual spectrometer experimental setup will not fill these requirements, since narrow angle illumination and collection is used; however, a microbeam condensor with suitable corrections or an integrating sphere can approximate the proper conditions.

NONISOTHERMAL LAYERS

In order to be useful in heat-transfer calculations, this theory must be extended to nonisothermal situations. This can be done (as is shown by Hamaker as well as Schuster) if in each radiation equation a term is added representing the amount of energy emitted by the infinitesimal region. This is $\epsilon E\, dx$ where ϵ is the emissivity and E is the black-body radiation at the temperature at x. Making use of Kirchhoff's law, this term becomes $aE\, dx$ where a is the previously defined absorption coefficient. An additional, heat balance, equation is now needed expressing the fact that heat is neither accumulated nor produced within the body:

$$\frac{k\, d^2T}{dx^2} + a(I + J) = 2aE \tag{32}$$

where k is the lattice thermal conductivity. The first term on the left side represents the heat accumulated by conduction; the second term is the heat absorbed from the radiation, and the sum of these equals the heat loss by radiation (the term on the right).

The total black-body radiation is given by the Stefan-Boltzmann equation:

$$E = \sigma' n^2 T^4 \tag{33}$$

where σ' is the Stefan-Boltzmann radiation constant and T is the absolute temperature. If the temperature is high and the temperature gradient not too large, then E may be represented by

$$E = E_o + b(T - T_o) \tag{34}$$

where

$$b = 4\sigma' n^2 T_o^3 \tag{35}$$

T_o is a temperature close to the actual temperature, and E_o is the corresponding total radiation. When the above equation holds, the temperature may be fixed equally as well by E as by T and, since this simplifies matters, E rather than T has been retained in the equations. The set of simultaneous differential equations is then

$$\frac{dI}{dx} = -(a + s)I + sJ + aE \tag{36}$$

$$\frac{dJ}{dx} = (a + s)J - sJ - aE \tag{37}$$

$$\frac{k}{b}\frac{d^2E}{dx^2} + a(I + J) = 2aE \tag{38}$$

Hamaker shows that the complete general solution of these equations is:

$$I = A(1 - \beta)e^{\sigma x} + B(1 + \beta)e^{-\sigma x} + C(\sigma x - \beta) + F \qquad (39)$$

$$J = A(1 + \beta)e^{\sigma x} + B(1 - \beta)e^{-\sigma x} + C(\sigma x - \beta) + F \qquad (40)$$

$$E = -A\kappa e^{\sigma x} - B\kappa e^{-\sigma x} + C\sigma x + F \qquad (41)$$

where

$$\sigma = + \left| \sqrt{\frac{2ab}{k} + a(a + 2s)} \right| = \sigma_o \sqrt{(1 + \kappa)} \qquad (42)$$

$$\beta = \frac{\sigma}{a + 2s} \qquad (43)$$

$$\kappa = \frac{2b}{k(a + 2s)} = \frac{2b\beta}{k\sigma} \qquad (44)$$

and the proper n^2 term which does not appear in Hamaker's work has been introduced here.

To illustrate how this theory might be used the particular solutions will be derived for a layer receiving radiation at both surfaces, and where heat is being conducted away from the surfaces. The amount of heat being conducted away from the surface must equal that conducted to the surface in the solid giving one boundary condition at each surface. The other two boundary conditions are supplied by the radiation interchange at the surface. The temperature (particularly at the surfaces) and the emitted fluxes will be solved for.

Immediately below the front surface $x = 0$, the forward flux I_o is equal to that part of the incident flux I_i which is not reflected $((1 - \rho_o)I_i)$ plus the amount of the backward flux at this surface J_o which was reflected $(\rho_i J_o)$. Therefore

$$I_o = (1 - \rho_o)I_i + \rho_i J_o \qquad (45)$$

or (substituting from eqs. (39) and (40)):

$$A(1 - \beta) + B(1 + \beta) - C\beta + F = (1 - \rho_o)I_i + \rho_i A(1 + \beta) + \rho_i B(1 - \beta) + \rho_i C\beta + F\rho_i \qquad (46)$$

Similarly, immediately below the back surface $x = D$, the backward flux J_D is composed of the part of the incident flux on this surface J_i which is transmitted $((1 - \rho_o)J_i)$ plus the part of the forward flux at this surface which is reflected $(\rho_i I_D)$, and

$$J_D = (1 - \rho_o)J_i + \rho_i I_D \tag{47}$$

or

$$A(1 + \beta)e^{\sigma D} + B(1 - \beta)e^{-\sigma D} + C(\sigma D + \beta) + F$$
$$= (1 - \rho_o)J_i + \rho_i A(1 - \beta)e^{\sigma D} + B\rho_i(1 + \beta)e^{-\sigma D} + \rho_i C(\sigma D - \beta) + \rho_i F \tag{48}$$

If we define η as being equal to the gradient at the surface times b,

$$\eta \equiv b\left(\frac{dT}{dx}\right)_{surface} \equiv \left(\frac{dE}{dx}\right)_{surface} \tag{49}$$

or if the heat is conducted away by a gas,

$$\eta \equiv \left(\frac{dE}{dx}\right)_{surface} = \frac{-bQ_g}{k} \tag{50}$$

where Q_g is the heat being conducted (or convected) away by the gas. Then, since (by differentiating eq. (41))

$$\frac{dE}{dx} = -A\kappa\sigma e^{\sigma x} + B\kappa\sigma e^{-\sigma x} + C\sigma \tag{51}$$

the other two boundary conditions are

$$\eta = \left(\frac{dE}{dx}\right)_o \tag{52}$$

$$\eta = -A\kappa\sigma + B\kappa\sigma + C\sigma \tag{53}$$

and

$$\eta = \left(\frac{dE}{dx}\right)_D \tag{54}$$

$$\eta = -A\kappa\sigma e^{\sigma D} + B\kappa\sigma e^{-\sigma D} + C\sigma \tag{55}$$

The four simultaneous equations (46), (48), (53), and (55) are then solved for the constants A, B, C, and F where it has been found convenient to define a function consisting of the denominator: Let

$$2(1 - \rho_i)(\cosh \sigma D - 1) + [2\beta(1 + \rho_i)(1 + \kappa) + \kappa\sigma D(1 - \rho_i)]\sinh \sigma D \equiv \text{etc} \tag{56}$$

Then

$$A = \frac{(e^{-\sigma D} - 1)\left\{\sigma(1 - \rho_o)(I_i - J_i) + \eta[2\beta(1 + \rho_i) + \sigma D(1 - \rho_i)]\right\}}{2\sigma \text{ etc}} \tag{57}$$

$$B = \frac{(e^{\sigma D} - 1)\left\{\sigma(1 - \rho_o)(I_i - J_i) + \eta[2\beta(1 + \rho_i) + \sigma D(1 - \rho_i)]\right\}}{2\sigma \text{ etc}} \tag{58}$$

$$C = \frac{-2\kappa\sigma(1 - \rho_o)(I_i - J_i)\sinh \sigma D + 4\eta[(1 - \rho_i)(\cosh \sigma D - 1) + \beta(1 + \rho_i)\sinh \sigma D}{2\sigma \text{ etc}} \tag{59}$$

$$F = \frac{\begin{array}{l} 2\sigma n^2 I_i\left\{(1 - \rho_i)(\cosh \sigma D - 1) + [\beta(1 + \rho_i)(1 + \kappa) + \kappa\sigma D(1 - \rho_i)]\sinh \sigma D\right\} \\ + 2\sigma n^2 J_i[(1 - \rho_i)(\cosh \sigma D - 1) + \beta(1 + \rho_i)(1 + \kappa)\sinh \sigma D] \\ - 2\sigma\eta D[(1 - \rho_i)(\cosh \sigma D - 1) + \beta(1 + \rho_i)\sinh \sigma D] \end{array}}{2\sigma \text{ etc}} \tag{60}$$

Introducing these constants into equations (39), (40), and (41) makes it possible now to find the fluxes and temperature at any point in terms of the incident radiant and thermal fluxes. They are

$$I = \frac{\begin{array}{l} \sigma(1 - \rho_o)(I_i - J_i)[e^{\sigma x}(e^{-\sigma D} - 1)(1 - \beta) + (e^{\sigma D} - 1)(1 + \beta)e^{-\sigma x} \\ - 2\kappa(\sigma x - \beta)\sinh \sigma D] + \eta\left\{[2\beta(1 + \rho_i) \right. \\ + \sigma D(1 - \rho_i)][e^{\sigma x}(e^{-\sigma D} - 1)(1 - \beta) + e^{-\sigma x}(e^{\sigma D} - 1)(1 + \beta)] \\ \left. + 4(\sigma x - \beta)[\beta(1 + \rho_i)\sinh \sigma D + (1 - \rho_i)(\cosh \sigma D - 1)]\right\} \end{array}}{2\sigma \text{ etc}} + F \tag{61}$$

$$J = \frac{\begin{array}{l} \sigma(1 - \rho_o)(I_i - J_i)[e^{\sigma x}(e^{-\sigma D} - 1)(1 + \beta) + e^{-\sigma x}(e^{\sigma D} - 1)(1 - \beta) \\ - 2\kappa(\sigma x + \beta)\sinh \sigma D] + \eta\left\{[2\beta(1 + \rho_i) \right. \\ + \sigma D(1 - \rho_i)][e^{\sigma x}(e^{-\sigma D} - 1)(1 + \beta) + e^{-\sigma x}(e^{\sigma D} - 1)(1 - \beta)] \\ \left. + 4(\sigma x + \beta)[\beta(1 + \rho_i)\sinh \sigma D + (1 - \rho_i)(\cosh \sigma D - 1)]\right\} \end{array}}{2\sigma \text{ etc}} + F \tag{62}$$

$$E = \frac{\begin{array}{l}-\kappa\sigma(1-\rho_o)(I_i - J_i)[e^{\sigma x}(e^{-\sigma D}-1) + e^{-\sigma x}(e^{\sigma D}-1) + 2\sigma x \sinh \sigma D] \\ - \eta\{\kappa[2\beta(1+\rho_i) + \sigma D(1-\rho_i)][e^{\sigma x}(e^{-\sigma D}-1) + e^{-\sigma x}(e^{\sigma D}-1)] \\ - 4\sigma x[\beta(1+\rho_i)\sinh \sigma D + (1-\rho_i)(\cosh \sigma D - 1)]\}\end{array}}{2\sigma \text{ etc}} + F \qquad (63)$$

These are again not measurable quantities. The quantities desired are the fluxes emitted at each surface and the temperatures at the surfaces. At the back of the layer, $x = D$, the flux emitted in the forward direction (here denoted by I_e) is equal to the fraction of the forward flux immediately under this surface I_D which is not reflected at this surface $(1 - \rho_i)I_D$ plus the fraction of the incident radiation on this surface which is reflected into the forward direction $\rho_o J_i$, or

$$I_e = (1 - \rho_i)I_D + \rho_o J_i \qquad (64)$$

Similarly (where J_e is the flux emitted in the backward direction at the front surface),

$$J_e = (1 - \rho_i)J_o + \rho_o I_i \qquad (65)$$

Finally, the energy equivalent of the temperature at the surfaces is found by substituting $x = 0$ and $x = D$ into equation (63). Actually in the constant b, the n^2 term should be the index of refraction of the material in which the particular quantity is measured. In order to keep the notation consistent in this part, the n^2 term will be kept in the constant b but the energy equivalent of temperature measured outside the sample (here denoted by E_{ao} and E_{aD}) will be divided by n^2, so that the numerical results will be correct. Before the results are set down, it is desirable to define the following functions since most of the equations are symmetrical.

Let

$$\text{etc} \equiv 2(1-\rho_i)(\cosh \sigma D - 1) + [2\beta(1+\rho_i)(1+\kappa) + \kappa\sigma D(1-\rho_i)]\sinh \sigma D \qquad (66)$$

$$f_1 \equiv \frac{2\beta(1-\rho_o)(1+\kappa)\sinh \sigma D}{\text{etc}} \qquad (67)$$

$$f_2 \equiv \frac{2(1-\rho_i)(\cosh \sigma D - 1) + [2\beta(\rho_o+\rho_i)(1+\kappa) + \kappa\sigma D(1-\rho_i)]\sinh \sigma D}{\text{etc}} \qquad (68)$$

$$f_3 \equiv \frac{(1-\kappa)(1-\rho_i)(\cosh \sigma D - 1) + [\beta(1+\rho_i)(1+\kappa) + \kappa\sigma D(1-\rho_i)]\sinh \sigma D}{\text{etc}} \qquad (69)$$

$$f_4 \equiv \frac{(1+\kappa)[(1-\rho_i)(\cosh \sigma D - 1) + \beta(1+\rho_i)\sinh \sigma D]}{\text{etc}} \tag{70}$$

$$f_5 \equiv \frac{(1-\rho_i)[-2(\cosh \sigma D - 1) + \sigma D \sinh \sigma D]}{\text{etc}} \tag{71}$$

$$f_6 \equiv \frac{[\sigma D(1+\kappa)(1-\rho_i) + 2\beta\kappa(1+\rho_i)](\cosh \sigma D - 1) + \beta\sigma D(1-\rho_i)\sinh \sigma D}{n^2 \text{ etc}} \tag{72}$$

Then the desired terms are (where Q_{-g} is the negative of the heat removed by conduction, and, therefore, $\eta \equiv bQ_{-g}/k$):

$$I_e = f_1 I_i + f_2 J_i + f_5 \kappa Q_{-g} \tag{73}$$

$$J_e = f_2 I_i + f_1 J_i - f_5 \kappa Q_{-g} \tag{74}$$

$$E_{ao} = f_3 I_i + f_4 J_i - f_6 \frac{\kappa}{2\beta} Q_{-g} \tag{75}$$

$$E_{aD} = f_4 I_i + f_3 J_i + f_6 \frac{\kappa}{2\beta} Q_{-g} \tag{76}$$

CONCLUSIONS AND DISCUSSION

The previous section illustrates how analytical expressions can be obtained to describe radiation transfer through scattering and absorbing nonisothermal layers. The actual choice of the independent and dependent variables is arbitrary; for instance, it is also possible to specify the surface temperatures and perhaps assume no incident fluxes, then solve for the emitted fluxes and the surface gradients necessary to maintain the given situation. Proper boundary conditions for several other cases are given in Hamaker's paper while reference 11 shows some calculations for a semitransparent layer on a metal. The latter paper also discusses the nature of the gradient changes at the interfaces, as well as the considerations that occur when the layer gets thin. Another application is given in reference 12 where radiation heat transfer through powders is treated by using a model of a system of layers through which the radiant transfer has been calculated. Whereas this simplified system allows one to calculate analytic expressions for radiant transfer in very complex situations, some of the limitations of the method should be noted. Even with perfectly diffuse radiation incident on a sample, the assumption of diffuse radiation right below a surface where the index of refraction is increasing is only an approximation, since the radiation will be brought into a narrower solid angle, as the result of refraction at the interface. It is assumed that the radiation is rapidly rediffused due to scattering. This treatment is, of course, invalid where the scattering centers are so close together that phase effects must be taken into account and coherent scattering occurs. Also, the method breaks down when the layers become so thin that their properties change; that is, they can no longer be considered to be homogeneous. Illustrations of this would be where the pores might be relatively large compared to the sample layer thickness. Finally, it should be noted that the absorption and scattering coefficients measured or calculated here from diffuse radiation measurements and calculations are not the same as would be measured by narrow angle measurements. The narrow angle measurements measure changes in the image forming part of the radiation only. The absorption coefficient defined here will actually be a function of the scattering coefficient, since the scattering coefficient will determine the actual path length through the sample and therefore the total amount of absorption.

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